New Possibility for a Unified Field Theory of Gravitation and Electricity

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Some days ago I explained in a short note in these reports, how by using a n-bein field a geometric theory can be constructed that is based on the notion of a Riemann-metric and distant parallelism. I left open the question if this theory could serve for describing physical phenomena. In the meantime I discovered that this theory - at least in first approximation – yields the field equations of gravitation and electromagnetism in a very simple and natural manner. Thus it seems possible that this theory will substitute the theory of general relativity in its original form.
The introduction of this theory has as a consequence the existence of a straight line, that means a line of which all elements are parallel to each other.[1] Naturally, such a line is not identical with a geodesic. Furthermore, contrarily to the actual theory of relativity, the notion of relative rest of two mass points exists (parallelism of two line elements that belong to two different world lines).
In order to apply the general theory in the implemented form to the field theory, one has to set the following conventions:

1. The dimension is 4 ($n = 4$).
2. The fourth local component A_a (a=4) of a vector is purely imaginary, and so are the components of the 4th bein of the 4-bein,[2] and also the quantities h_4^ν and $h_{\nu 4}$[3]. Of course, all the coefficients of $\gamma_{\mu\nu}$ $(= h_{\mu a} h_{\nu a})$ become then real. Thus, we choose the square of the modulus of a timelike vector to be negative.

1 The assumed basic field law

For the variation of the field potentials $h_{\mu a}$ (or h_α^μ) to vanish on the boundary of a domain the variation of the Hamiltonian integral should vanish:

$$\delta\{\int \mathcal{H} d\tau\} = 0, \tag{1}$$

$$\mathcal{H} = h\ g^{\mu\nu}\ \Lambda_{\mu\beta}^{\ \ \alpha}\ \Lambda_{\nu\alpha}^{\ \ \beta}, \tag{1a}$$

with the quantities $h(=\mid h_{\mu\alpha} \mid)$, $g^{\mu\nu}$, $\Lambda_{\mu\nu}^{\ \ \alpha}$ defined in the eqns. (9), (10) loc.it.[4]
The field h should describe both the electrical and gravitational field. A 'pure gravitational' field means that in addition to eqn. (1) being satisfied the quantities

$$\phi_\mu = \Lambda_{\mu\alpha}^{\ \ \alpha} \tag{2}$$

[1] tr. note: such lines are nowadays called autoparallels.
[2] tr.note: 4-bein (tetrad, from German 'bein' = leg) has become a common expression in differential geometry)
[3] Instead of this one could define the square of the length of the local vector $A_1^2 + A_2^2 + A_3^2 - A_4^2$ and introduce Lorentz-transformations instead of rotations of the local n-bein. In that case all the h would become real, but one would loose the direct connection to the formulation of the general theory.
[4] tr. note: this refers to the session report of June 7th, 1928.

vanish, which is a covariant and rotation invariant restriction.[5]

2 The field law in first approximation

If the manifold is the MINKOWSKI-world of the theory of special relativity, one may choose the coordinate system in a way that $h_{11} = h_{22} = h_{33} = 1, h_{44} = j$ $(= \sqrt{-1})$ holds and the other $h_{\mu\alpha}$ vanish. This system of values for $h_{\mu\alpha}$ is a little inconvenient for calculations. Therefore in this paragraph for calculations we prefer to assume the x_4-coordinate to be purely imaginary; then, the MINKOWSKI-world (absence of any field for a suitable choice of coordinates) can be described by

$$h_{\mu\alpha} = \delta_{\mu\alpha} \ldots \tag{3}$$

The case of infinitely weak fields can be described purposively by

$$h_{\mu\alpha} = \delta_{\mu\alpha} + k_{\mu\alpha}, \ldots \tag{4}$$

whereby the $k_{\mu\alpha}$ are small values of first order. While neglecting quantities of third and higher order one has to replace (1a) with respect to (10) and (7a) loc. it. by

$$\mathcal{H} = \frac{1}{4}(\frac{\partial k_{\mu\alpha}}{\partial x_\beta} - \frac{\partial k_{\beta\alpha}}{\partial x_\mu})(\frac{\partial k_{\mu\beta}}{\partial x_\alpha} - \frac{\partial k_{\alpha\beta}}{\partial x_\mu}). \tag{1b}$$

By performing the variation one obtains the field equations valid in first approximation

$$\frac{\partial^2 k_{\beta\alpha}}{\partial x_\mu^2} - \frac{\partial^2 k_{\mu\alpha}}{\partial x_\mu \partial x_\beta} + \frac{\partial^2 k_{\alpha\mu}}{\partial x_\mu \partial x_\beta} - \frac{\partial^2 k_{\beta\mu}}{\partial x_\mu \partial x_\alpha} = 0. \ldots \tag{5}$$

This are 16 equations[6] for the 16 quantities $k_{\alpha\beta}$. Our task is now to see if this system of equations contains the known laws of gravitational and the electromagnetical field. For this purpose we introduce in (5) the $g_{\alpha\beta}$ and the ϕ_α instead of the $k_{\alpha\beta}$. We have to define

$$g_{\alpha\beta} = h_{\alpha a} h_{\beta a} = (\delta_{\alpha a} + k_{\alpha a})(\delta_{\beta a} + k_{\beta a})$$

or in first order

$$g_{\alpha\beta} - \delta_{\alpha\beta} = \overline{g_{\alpha\beta}} = k_{\alpha\beta} + k_{\beta\alpha}. \ldots \tag{6}$$

From (2) one obtains further the quantities of first order, precisely

$$2\ \phi_\alpha = \frac{\partial k_{\alpha\mu}}{\partial x_\mu} - \frac{\partial k_{\mu\mu}}{\partial x_\alpha}. \ldots \tag{2a}$$

By exchanging α and β in (5) and adding the thus obtained structuring of (5) at first one gets

$$\frac{\partial^2 \overline{g_{\alpha\beta}}}{\partial x_\mu^2} - \frac{\partial^2 k_{\mu\alpha}}{\partial x_\mu \partial x_\beta} - \frac{\partial^2 k_{\mu\beta}}{\partial x_\mu \partial x_\alpha} = 0.$$

If to this equation the two equations

$$-\frac{\partial^2 k_{\alpha\mu}}{\partial x_\mu \partial x_\beta} + \frac{\partial^2 k_{\mu\mu}}{\partial x_\alpha \partial x_\beta} = -2\frac{\partial \phi_\alpha}{\partial x_\beta}$$
$$-\frac{\partial^2 k_{\beta\mu}}{\partial x_\mu \partial x_\alpha} + \frac{\partial^2 k_{\mu\mu}}{\partial x_\alpha \partial x_\beta} = -2\frac{\partial \phi_\beta}{\partial x_\alpha},$$

[5] There is still a certain ambiguity in interpreting, because one could characterize the gravitational field by the vanishing of $\frac{\partial \phi_\mu}{\partial x_\nu} - \frac{\partial \phi_\nu}{\partial x_\mu}$ as well.

[6] Naturally, between the field equations there exist four identities due to the general covariance. In the first approximation treated here this is expressed by the fact that the divergence taken with respect to the index a of the l.h.s. of (5) vanishes identically.

are added, following from (2a), one obtains according to (6)

$$\frac{1}{2}(-\frac{\partial^2\overline{g_{\alpha\beta}}}{\partial x_\mu^2}+\frac{\partial^2\overline{g_{\mu\alpha}}}{\partial x_\mu \partial x_\beta}+\frac{\partial^2\overline{g_{\mu\beta}}}{\partial x_\mu \partial x_\alpha}-\frac{\partial^2\overline{g_{\mu\mu}}}{\partial x_\alpha \partial x_\beta}+)=\frac{\partial\phi_\alpha}{\partial x_\beta}+\frac{\partial\phi_\beta}{\partial x_\alpha}\dots. \tag{7}$$

The case of the absence of an electromagnetic field is characterized by the vanishing of ϕ_μ. In this case (7) is in first order equivalent to the equation

$$R_{\alpha\beta}=0$$

used as yet in the theory of general relativity ($R_{\alpha\beta}$ = contracted Riemann tensor). *With the help of this it is proved that our new theory yields the law of a pure gravitational field in first approximation correctly*.

By differentiation of (2a) by x_α, one gets the equation

$$\frac{\partial\phi_\alpha}{\partial x_\alpha}=0. \tag{8}$$

according to (5) and contraction over α and β. Taking into account that the l.h.s. $L_{\alpha\beta}$ of (7) fulfills the identity

$$\frac{\partial}{\partial x_\beta}(L_{\alpha\beta}-\frac{1}{2}\delta_{\alpha\beta}L_{\sigma\sigma})=0,$$

from (7) follows

$$\frac{\partial^2\phi_\alpha}{\partial^2 x_\beta}+\frac{\partial^2\phi_\beta}{\partial x_\alpha \partial x_\beta}-\frac{\partial}{\partial x_\alpha}(\frac{\partial\phi_\sigma}{\partial x_\sigma})=0$$

or

$$\frac{\partial^2\phi_\alpha}{\partial^2 x_\beta}=0.\dots \tag{9}$$

The equations (8) and (9) are, as it is well known, equivalent to Maxwell's equations for empty space. *The new theory thus also yields* Maxwell*'s equations* in first approximation.

According to this theory, the separation of the gravitational and electromagnetic field seems arbitrary however. Furthermore, it is clear that the eqns. (5) state more than the eqns. (7), (8) and (9) together. After all it is remarkable that the electric field does not enter the field equations quadratically.

Note added in proof. One obtains very similar results by starting with the Hamilton function

$$\mathcal{H}=h\ g_{\mu\nu}g^{\alpha\sigma}g^{\beta\tau}\ \Lambda_{\alpha\beta}^{\ \ \mu}\Lambda_{\sigma\tau}^{\ \ \nu}.$$

Thus for the time being there remains a certain insecurity regarding the choice of $\mathcal{H}$.