Riemannian Geometry with Maintaining the Notion of Distant Parallelism

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Riemannian Geometry has led to a physical description of the gravitational field in the theory of general relativity, but it did not provide concepts that can be attributed to the electromagnetic field. Therefore, theoreticians aim to find natural generalizations or extensions of riemannian geometry that are richer in concepts, hoping to arrive at a logical construction that unifies all physical field concepts under one single leading point. Such endeavors brought me to a theory which should be communicated even without attempting any physical interpretation, because it can claim a certain interest just because of the naturality of the concepts introduced therein.
Riemannian geometry is characterized by an Euclidean metric in an infinitesimal neighborhood of any point P. Furthermore, the absolute values of the line elements which belong to the neighborhood of two points P and Q of finite distance can be compared. However, the notion of parallelism of such line elements is missing; a concept of direction does not exist for the finite case. The theory outlined in the following is characterized by introducing - beyond the Riemannian metric- the concept of 'direction', 'equality of directions' or 'parallelism' for finite distances. Therefore, new invariants and tensors will arise besides those known in Riemannian geometry.

1 n-bein field and metric

Given an arbitrary point P of the n-dimensional continuum, let's imagine an orthogonal n-bein of n unit vectors that represents a local coordinate system. A_a are the components of a line element or another vector with respect to this local system (n-bein). Besides that, we introduce a Gaussian coordinate system of the x^ν for describing a finite domain. Let A^ν be the components of a vector (A) with respect to the latter, and ${h^\nu}_a$ the ν-components of the unit vectors forming the n-bein. Then, we have[1]

$$A^\nu = {h^\nu}_a A_a \ldots. \tag{1}$$

One obtains the inversion of (1) by calling $h^\nu_{\nu a}$ the normalized subdeterminants of the ${h^\nu}_a$,

$$A_a = h_{\mu a} A^\mu \ldots. \tag{1a}$$

Since the infinitesimal sets are Euclidean,

$$A^2 = \sum A_a^2 = h_{\mu a}\ h_{\nu a} A^\mu A^\nu \ldots \tag{2}$$

holds for the modulus A of the vector (A).
Therefore, the components of the metric tensor appear in the form

$$g_{\mu\nu} = h_{\mu a}\ h_{\nu a}, \ldots \tag{3}$$

[1]We assign Greek letters to the coordinate indices and Latin ones to the bein indices.

whereby the sum has to be taken over a. For a fixed a, the h^μ_a are the components of a contravariant vector. Furthermore, the following relations hold:

$$h_{\mu a}\ h^\nu_a = \delta^\nu_\mu \ldots \tag{4}$$

$$h_{\mu a}\ h^\mu_b = \delta_{ab}, \ldots \tag{5}$$

with $\delta = 1$ if the indices are equal, and $\delta = 0$, if not. The correctness of (4) and (5) follows from the above definition of the $h_{\mu a}$ as the normalized subdeterminants of the h^μ_a. The vector property of $h_{\mu a}$ follows conveniently from the fact that the l.h.s. and therefore, the r.h.s. of (1a) as well, are invariant for any coordinate transformation and for any choice of the vector (A). The n-bein field is determined by n^2 functions h^μ_a, whereas the Riemannian metric is determined just by $\frac{n(n+1)}{2}$ quantities. According to (3), the metric is determined by the n-bein field but not vice versa.

2 Teleparallelism and rotation invariance

By postulating the existence of the n-bein field (in every point) one expresses implicitly the existence of a Riemannian metric and distant parallelism. (A) and (B) being two vectors in the points P and Q which have the same local coordinates with respect to their n-beins (that means $A_a = B_a$), then have to be regarded as equal (because of (2)) and as 'parallel'.
If we take the metric and the teleparallelism as the essential, i.e. the objective meaningful things, then we realize that the n-bein field is not yet fully determined by these settings. Yet metric and teleparallelism remain intact, if we substitute the n-beins of all points of the continuum with such n-beins that were derived out of the original ones by the rotation stated above. We denote this substitutability of the n-bein field as rotational invariance and establish: Only those mathematical relations that are rotational invariant can claim a real meaning.
Thus by keeping the coordinate system fixed, and a given metric and parallel connection, the h^μ_a are not yet fully determined; there is a possible substitution which corresponds to the rotation invariance

$$A^*_a = d_{a\ m} A_m \cdots, \tag{6}$$

whereby $d_{a\ m}$ is chosen orthogonal and independent of the coordinates. (A_a) is an arbitrary vector with respect to the local system, (A^*_a) the same vector with respect to the rotated local system. According to (1a), and using (6), it follows

$$h^*_{\mu a} A^\mu = d_{am} h_{\mu m} A^\mu$$

or

$$h^*_{\mu a} = d_{am} h_{\mu m}, \ldots \qquad (6a)$$

whereby

$$d_{am}\ d_{bm} = d_{ma}\ d_{mb} = \delta_{ab}, \ldots \qquad (6b)$$

$$\frac{\partial d_{am}}{\partial x^\nu} = 0. \ldots \qquad (6c)$$

Now the postulate of rotation invariance tells us that among the relations in which the quantities h appear, only those may be seen as meaningful, which are transformed into h^* of equal form, if h^* is introduced by eqns. (6). In other words: n-bein fields which are related by locally equal rotations are equivalent.
The rule of infinitesimal parallel transport of a vector from point (x^ν) to a neighboring point $(x^\nu + dx^\nu)$ is obviously characterized by

$$d\ A_a = 0 \ldots, \tag{7}$$

that means by the equation

$$0 = d(h_{\mu a} A^\mu) = \frac{\partial h_{\mu a}}{\partial x^\sigma} A^\mu dx^\sigma + h_{\mu a} dA^\mu = 0$$

Mulitplicated by h_a^ν this equation becomes considering (5)

$$dA^\nu = -\Delta^\nu_{\mu\sigma} A^\mu dx^\sigma \qquad (7a)$$

with

$$\Delta^\nu_{\mu\sigma} = h_a^\nu \frac{\partial h_{\mu a}}{\partial x^\sigma}.$$

This law of parallel transport is rotation invariant and not symmetric with respect to the lower indices of the quantities $\Delta^\nu_{\mu\sigma}$. If one transports the vector (A) now according to this law along a closed path, the vector remains unaltered; this means, that the Riemannian tensor

$$R^i_{k,lm} = -\frac{\partial \Delta^i_{kl}}{\partial x^m} + \frac{\partial \Delta^i_{km}}{\partial x^l} + \Delta^i_{\alpha l}\Delta^\alpha_{km} - \Delta^i_{\alpha m}\Delta^\alpha_{kl}$$

built from the connection coefficients vanishes according to (7a), which can be verified easily. Besides this law of parallel transport there is that (nonintegrable) symmetric transport law due to the Riemannian metric (2) and (3). As is generally known, it is given by the equations

$$\bar{d}A^\nu = -\Gamma^\nu_{\mu\tau} A^\mu dx^\tau \qquad (8)$$

$$\Gamma^\nu_{\mu\tau} = \frac{1}{2} g^{\nu\alpha}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\tau} + \frac{\partial g_{\tau\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\tau}}{\partial x^\alpha}\right).$$

According to (3), the $\Gamma^\nu_{\mu\tau}$ are expressed by the quantities h of the n-bein fields. Thereby one has to keep in mind that

$$g^{\mu\nu} = h^\mu_\alpha h^\nu_\alpha \ldots. \qquad (9)$$

Because of this setting and due to (4) and (5) the equations

$$g^{\mu\lambda} g_{\nu\lambda} = \delta^\mu_\nu$$

are fulfilled which define the $g^{\mu\lambda}$ calculated from the $g_{\mu\lambda}$. This transport law based on metric only is obviously rotation invariant in the above sense.

3 Invariants and covariants

On the manifold we are considering, besides the tensors and invariants of RIEMANN-geometry which contain the quantities h only in the combination (3), other tensors and invariants exist, among which we will have a look at the simplest ones only.
If one starts with a vector (A^ν) in the point x^ν, with the shifts d and $\bar{d}$, the two vectors

$$A^\nu + dA^\nu$$

and

$$A^\nu + \bar{d}A^\nu$$

are produced in the neighboring point $(x + dx^\nu)$. Thus the difference

$$dA^\nu - \bar{d}A^\nu = (\Gamma^\nu_{\alpha\beta} - \Delta^\nu_{\alpha\beta}) A^\alpha dx^\beta$$

has vector character as well. Therefore,

$$(\Gamma^\nu_{\alpha\beta} - \Delta^\nu_{\alpha\beta})$$

is a tensor, and also its skewsymmetric part

$$\frac{1}{2}(\Delta^\nu_{\alpha\beta} - \Delta^\nu_{\beta\alpha}) = \Lambda^\nu_{\beta\alpha} \ldots \qquad (10)$$

The fundamental meaning of this tensor in the theory developed here results from the following: If this tensor vanishes, then the continuum is Euclidean. Namely, if

$$0 = 2\Lambda^{\nu}_{\alpha\beta} = h^{\nu}_{a}(\frac{\partial h_{\alpha a}}{\partial x^{\beta}} - \frac{\partial h_{\beta a}}{\partial x^{\alpha}}),$$

holds, then by multiplication with $h_{\nu b}$ follows

$$0 = \frac{\partial h_{\alpha b}}{\partial x^{\beta}} - \frac{\partial h_{\beta b}}{\partial x^{\alpha}}.$$

However, one may assume

$$h_{ab} = \frac{\partial \psi_b}{\partial x^{\alpha}}.$$

Therefore the field is derivable from n scalars ψ_b. We now choose the coordinates according to the equation

$$\psi_b = x^b$$

Then, due to (7a) all the $\Delta^{\nu}_{\beta\alpha}$ vanish, and the $h_{\mu a}$ and the $g_{\mu\nu}$ are constant.—
Since the tensor[2] $\Lambda^{\nu}_{\beta\alpha}$ is formally the simplest one admitted by our theory, this tensor shall be used as a starting point for characterizing such a continuum, and not the more complicated Riemannian curvature tensor. The most simple quantities which come in mind are the vector

$$\Lambda^{\alpha}_{\mu\alpha}$$

and the invariants

$$g^{\mu\nu}\Lambda^{\alpha}_{\mu\beta}\Lambda^{\beta}_{\nu\alpha} \text{ and } g_{\mu\nu}g^{\alpha\sigma}g^{\beta\tau}\Lambda^{\mu}_{\alpha\beta}\Lambda^{\nu}_{\sigma\tau}$$

From one of the latter ones (actually, from a linear combination of it), after multiplication with the invariant volume element

$$h\ d\tau,$$

(whereby h means the determinant $\mid h_{\mu\alpha} \mid$, $d\tau$ the product $dx_1...dx_n$), an invariant integral J, may be built. The setting

$$\delta J = 0$$

then provides 16 differential equations for the 16 quantities $h_{\mu\alpha}$.
If laws with relevance to physics can be derived from this, shall be investigated later.— It clarifies things, to compare Weyl's modification of the Riemannian theory to the one presented here:

> Weyl: no comparison at a distance, neither of the absolute values, nor of directions of vectors.
> Riemann: comparison at a distance for absolute values of vectors, but not of directions of vectors.
> Present theory: comparison of both absolute values and directions of vectors at a distance.[3]

[2]tr. note: this is called torsion tensor in the literature.
[3]tr. note: This is the origin of the name *distant parallelism* as a synonym for *absolute parallelism* or *teleparallelism*, in German *Fernparallelismus*.