Unified Field Theory Based on Riemannian Metrics and Distant Parallelism

Albert Einstein
translation by A. Unzicker and T. Case

Math. Annal. 102 (1930), pp. 685-697

In the present work I would like to describe a theory I have been working on for a year; it will be exposed in a manner that it can be understood comfortably by everyone who is familiar with the theory of general relativity. The following exposure is necessary, because due to coherences and improvements found in the meantime reading the earlier work would be a useless loss of time. The topic is presented in a way that seems most serviceable for comfortable access. I learned, especially with the help of Mr. Weitzenböck and Mr. Cartan, that the dealing with the continua we are talking about is not new. Mr. Cartan kindly wrote an essay about the history of the relevant mathematical topic in order to complete my paper; it is printed right after this paper in the same review. I would also like to thank Mr. Cartan heartily at this point for his valuable contribution. The most important and undisputable new result of the present work is the finding of the most simple field laws that can be applied to a Riemannian manifold with distant parallelism. I am only going to discuss their physical meaning briefly.

1 The structure of the continuum

Since the number of dimensions has no impact on the following considerations, we suppose a n-dimensional continuum. To take into account the facts of metrics and gravitation we assume the existence of a Riemann-metric. In nature there also exist electromagnetic fields, which cannot be described by Riemannian metrics. This arouses the question: How can we complement our Riemannian spaces in a natural, logical way with an additional structure, so that the whole thing has a uniform character ?
The continuum is (pseudo-)Euclidean in the vicinity of every point P. In every point there exists a local coordinate system of geodesics (i.e. an orthogonal n-bein), in relation to which the theorem of Pythagoras is valid. The orientation of these n-beins is not important in a Riemannian manifold. We would now like to assume that these elementary Euclidean spaces are governed by still another direction law. We are also going to assume, that it makes sense to speak of a parallel orientation of all n-beins together, applying this to space structure like in Euclidean geometry (which would be senseless in a space with metrical structure *only*).

In the following we are going to think of the orthogonal n-beins as being always in parallel orientation. The in its self arbitrary orientation of the local n-bein in one point P then determines the orientation of the local n-beins in all points of the continuum uniquely. Our task now is to set up the most simple restrictive laws which can be applied to such a continuum. Doing so, we hope to derive the general laws of nature, as the previous theory of general relativity tried this for gravitation by applying a purely metrical space structure.

2 Mathematical description of the space structure

The local n-bein consists of n orthogonal unit vectors with components ${h_s}^\nu$ with respect to any Gaussian coordinate system. Here as always a lower Latin index indicates the affiliation to a certain bein of the n-beins, a Greek index - due to its upper or lower position - the covariant or contravariant transformation character of the relevant entity with respect to a change of the Gaussian coordinate system. The general transformation property of the ${h_s}^\nu$ is the following. If all local systems or n-beins are twisted in the same manner, which is a correct operation of course, and a new Gaussian coordinate system is introduced at the same time, the following transformation law then exists in-between the new and old ${h_s}^\nu$

$$ {h_s}^{\nu'} = \alpha_{st} \frac{\partial x^{\nu'}}{\partial x^\alpha} {h_t}^\alpha, \tag{1} $$

whereas the constant coefficients α_{st} form an orthogonal system:

$$ \alpha_{sa}\alpha_{sb} = \alpha_{as}\alpha_{bs} = \delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases} \tag{2} $$

Without problems the transformation law (1) can be generalized onto objects which components bear an arbitrary number of local indices and coordinate indices. We call such objects tensors. Out of this the algebraic laws of tensors (addition, multiplication, contraction by Latin and Greek indices) follow immediately.

We call ${h_s}^\nu$ the components of the fundamental tensor. If a vector in the local system has components A_s, and the coordinates A^ν with respect to the Gaussian system, it follows out of the meaning of the ${h_s}^\nu$:

$$ A^\nu = {h_s}^\nu A_s \tag{3} $$

or – resolved with respect to the A_s–

$$ A_s = h_{s\nu} A^\nu \tag{4} $$

The tensorial character of the normalized subdeterminants $h_{s\nu}$ of the ${h_s}^\nu$ follows out of (4). $h_{s\nu}$ are the covariant components of the fundamental tensor. Between $h_{s\nu}$ and ${h_s}^\nu$ there are the relations

$$ h_{s\mu}{h_s}^\nu = {\delta_\mu}^\nu = \begin{cases} 1, & \text{if } \mu = \nu \\ 0, & \text{if } \mu \neq \nu \end{cases} \tag{5} $$

$$ h_{s\mu}{h_t}^\mu = \delta_{st} \tag{6} $$

Due to the orthogonality of the local system we obtain the absolute value of the vector

$$ A^2 = A_s^2 = h_{s\mu}\ h_{s\nu}\ A^\mu\ A^\nu = g_{\mu\nu}\ A^\mu\ A^\nu; \tag{6} $$

Therefore,

$$ g_{\mu\nu} = h_{s\mu}\ h_{s\nu} \tag{7} $$

are the coefficients of the metric.

The fundamental tensor allows (cfr. (3) and (4)) to transform local indices into coordinate indices and vice versa (by multiplication and contraction), so that it comes down to pure convention, with which type of tensors one likes to operate.

Obviously the following relations hold:

$$ A_\nu = h_{s\nu} A_s, \qquad (3a) $$

$$ A_s = {h_s}^\nu A_\nu. \qquad (4a) $$

Furthermore, we have the relation of determinants

$$g = |g_{\sigma\tau}| = |h_{\alpha\sigma}|^2 = h^2 \tag{8}$$

Therefore, the invariant of the volume element $\sqrt{g d\tau}$ takes the form $h d\tau$. To take into account the particular properties of time, it is most comfortable to set the x^4-coordinate (both local and general) of our 4-dimensional space-time continuum purely imaginary and also all tensor components with an odd number of indices 4.

3 Differential relations

Now we denote δ the change of the components of a vector or tensor during a 'parallel displacement' in the sense of Levi-Civita during the transition to a infinitely neighboring point of the continuum; now it follows out of the above

$$0 = \delta A_s = \delta(h_{s\alpha} A^\alpha) = \delta({h_s}^\alpha A_\alpha) \tag{9}$$

Resolving the brackets yields

$$h_{s\alpha}\ \delta A^\alpha + A^\alpha\ h_{s\alpha,\beta}\ \delta x^\beta = 0,$$
$${h_s}^\alpha\ \delta A_\alpha + A_\alpha\ {h_s}^\alpha{}_{,\beta}\ \delta x^\beta = 0,$$

whereas the colon indicates ordinary differentiation by x^β. Resolving of the equation yields

$$\delta A^\sigma = -A^\alpha \Delta_\alpha{}^\sigma{}_\beta\ \delta x^\beta, \tag{10}$$

$$\delta A_\sigma = A_\alpha \Delta_\sigma{}^\alpha{}_\beta\ \delta x^\beta, \tag{11}$$

whereby we set

$$\Delta_\alpha{}^\sigma{}_\beta = {h_s}^\sigma\ h_{s\alpha,\beta} = -h_{s\alpha}\ {h_s}^\sigma{}_{,\beta} \tag{12}$$[1]

(The last conversion is based on(5)).
This law of parallel displacement is – contrarily to Riemannian geometry – in general not symmetric. If it is, we have Euclidean geometry, because

$$\Delta_\alpha{}^\sigma{}_\beta - \Delta_\beta{}^\sigma{}_\alpha = 0$$

or

$$h_{s\alpha,\beta} - h_{s\beta,\alpha} = 0.$$

But then

$$h_{s\alpha} = \frac{\partial \psi_s}{\partial x_\alpha}$$

holds. If one chooses the ψ_s as new variables x'_s, we obtain

[1] tr. note: the connection Δ is nowadays usually denoted as Γ. Cfr. Schouten, Ricci Calculus (Springer, 1954), chap. III (1.2)

$$h_{s\alpha} = \delta_{s\alpha}, \tag{13}$$

proving the statement.

Covariant differentiation. The local components of a vector are invariant with respect to any coordinate transformation. Out of this follows immediately the tensorial character of the differential quotient

$$A_{s,\alpha}. \tag{14}$$

Because of (4a) this can be replaced by

$$(h_s{}^\sigma A{}_\sigma)_{,\alpha},$$

and the tensorial character of

$$h_s{}^\sigma A_{\sigma,\alpha} + A_\sigma \ h_s{}^\sigma{}_{,\alpha},$$

follows. Equally (after multiplication with $h_{s\tau}$) the tensorial character of

$$A_{\tau,\alpha} + A_\sigma \ h_s{}^\sigma{}_{,\alpha} \ h_{s\tau}$$

and of

$$A_{\tau,\alpha} - A_\sigma \ h_s{}^\sigma h_{s\tau,\alpha}$$

and (see (16)) of

$$A_{\tau,\alpha} - A_\sigma \Delta_\tau{}^\sigma{}_\alpha.$$ [2]

We call this covariant derivative ($A_{\tau;\alpha}$) of A_τ).
Therefore, we obtain the law of covariant differentiation

$$A_{\sigma;\tau} = A_{\sigma,\tau} - A_\alpha \Delta_\sigma{}^\alpha{}_\tau \tag{15}$$

Analogously, out of (3) follows the formula

$$A^\sigma{}_{;\tau} = A^\sigma{}_{,\tau} + A^\alpha \Delta_\alpha{}^\sigma{}_\tau. \tag{16}$$

The result is the law of covariant differentiation for arbitrary tensors. We illustrate this giving an example:

$$A_a{}^\sigma{}_{\tau;\rho} = A_a{}^\sigma{}_{\tau,\rho} + A_a{}^\alpha{}_\tau \Delta_\alpha{}^\sigma{}_\rho - A_a{}^\sigma{}_\alpha \Delta_\tau{}^\alpha{}_\rho. \tag{17}$$

By means of the fundamental tensor $h_s{}^\alpha$ we are allowed to transform local (Latin) indices in coordinate (Greek) indices, so we are free to favor the local or coordinate indices when formulating some tensor relations. The first approach is preferred by the Italian colleagues (Levi-Civita, Palatini), while I have preferably used coordinate indices.

Divergence. By contraction of the covariant differential quotient one obtains the divergence as in the absolute differential calculus based on metrics only. E.g., one gets the tensor

$$A_{\alpha\tau} = A_\alpha{}^\sigma{}_{\tau;\sigma}.$$

[2] tr. note: cfr. Schouten III, (1.3)

out of (21) by contraction of the indices σ and ρ.

In earlier papers I even introduced other divergence operators, but I do not accredit special significance to those any more.

Covariant differential quotients of the fundamental tensor.
One can easily find out of the formulas derived above, that the covariant derivatives and divergences of the fundamental tensor vanish. E.g. we have

$$h_s{}^\nu{}_{;\tau} \equiv h_s{}^\nu{}_{,\tau} + h_s{}^\alpha \Delta_\alpha{}^\nu{}_\tau \equiv \delta_{st}(h_t{}^\nu{}_\tau + h_t{}^\alpha \Delta_\alpha{}^\nu{}_{,\tau}) \tag{18}$$
$$\equiv h_s{}^\alpha (h_{t\alpha}\ h_t{}^\nu{}_{,\tau} + \Delta_\alpha{}^\nu{}_\tau) \equiv h_s{}^\alpha(-\Delta_\alpha{}^\nu{}_\tau + \Delta_\alpha{}^\nu{}_\tau) \equiv 0.$$

Analogously we can prove

$$h_s{}^\nu{}_\tau \equiv g^{\mu\nu}{}_{;\tau} \equiv g_{\mu\nu;\tau} \equiv 0. \tag{18a}$$

Likewise, the divergences $h_s{}^\nu{}_{;\nu}$ and $g^{\mu\nu}{}_{;\nu}$ obviously vanish.

Differentiation of tensor products. As it is apparent in the well-known differential calculus the covariant differential quotient of a tensor product can be expressed by the differential quotient of the factors. If $S_{\cdot}{}^{\cdot}$ and $T_{\cdot}{}^{\cdot}$ are tensors of arbitrary index character,

$$(S_{\cdot}{}^{\cdot}\ T_{\cdot}{}^{\cdot})_{;\alpha} = S_{\cdot}{}^{\cdot}{}_{;\alpha}\ T_{\cdot}{}^{\cdot} + T_{\cdot}{}^{\cdot}{}_{;\alpha}\ S_{\cdot}{}^{\cdot}. \tag{19}$$

follows. Out of this and out of the vanishing covariant differential quotient of the fundamental tensor it follows, that the latter may be interchanged with the differentiation symbol(;).

"Curvature". Out of the hypothesis of "distant parallelism" and out of equation (9) we obtain the integrability of the displacement law (10) and (11). Out of this follows

$$0 \equiv -\Delta_\kappa{}^\tau{}_{\lambda;\mu} \equiv -\Delta_\kappa{}^\tau{}_{\lambda,\mu} + \Delta_\kappa{}^\tau{}_{\mu,\lambda} + \Delta_\sigma{}^\tau{}_\lambda \Delta_\kappa{}^\sigma{}_\mu - \Delta_\sigma{}^\tau{}_\mu \Delta_\kappa{}^\sigma{}_\lambda. \tag{20}$$

In order to be expressed by the entities h ,the Δ's must comply to these conditions (cfr.(12)). Looking at (20), it is clear that the characteristic laws of the manifold in consideration here must be very different from the earlier theory. Though according to the new theory all tensors of the earlier theory exist, in particular the Riemannian curvature tensor calculated from the Christoffel symbols. But according to the new theory there are simpler and more elementary tensorial objects, that can be used for formulating the field laws.

The tensor Λ [3]. If we differentiate a scalar ψ twice covariantly, we obtain according to (15) the tensor

$$\phi_{,\sigma,\tau} - \phi_{,\alpha}\ \Delta_\sigma{}^\alpha{}_\tau.$$

From this follows at once the tensorial character of

$$\frac{\partial \phi}{\partial x_\alpha}(\Delta_\sigma{}^\alpha{}_\tau - \Delta_\tau{}^\alpha{}_\sigma).$$

Interchanging σ and τ a new tensor emerges and the subtraction yields the tensor

$$\Lambda_\sigma{}^\alpha{}_\tau = \Delta_\sigma{}^\alpha{}_\tau - \Delta_\tau{}^\alpha{}_\sigma.^4 \tag{21}$$

According to this theory there is a tensor containing the components $h_{\sigma\alpha}$ of the fundamental tensor and its first differential quotients only. We already proved that a vanishing fundamental tensor causes the validity of Euclidean geometry (cfr. (13)). Therefore, a natural law for such a continuum

[3]Cartans torsion tensor is nowadays usually denoted as T or S (Schouten)

[4]tr. note: cfr. Schouten III, (2.13)

will consist of conditions for this tensor.
By contraction of the tensor Λ we obtain

$$\phi_\sigma = \Lambda_{\sigma\ \alpha}^{\ \alpha}.^{5} \tag{22}$$

A vector which, as I suspected earlier, could take the part of the electromagnetic potential in the present theory, but ultimately I do not uphold this view.
Changing rule of differentiation. If a tensor $T_{\cdot}^{\ \cdot}$ is differentiated twice covariantly, the important rule holds

$$T_{\cdot\ ;\sigma;\tau}^{\ \cdot} - T_{\cdot\ ;\tau;\sigma}^{\ \cdot} \equiv -T_{\cdot\ ;\alpha}^{\ \cdot}\Lambda_{\sigma\ \tau}^{\ \alpha}. \tag{23}$$

Proof. If T is a scalar (tensor without Greek index), we obtain the proof without effort using (15). In this special case we will find the proof of the general theorem.
The first remark we would like to make is, that according to the theory discussed here parallel vector fields do exist. These vector fields have the same components in all local systems. If (a^a) or (a_a) is such a vector field, it satisfies the condition

$$a^a_{\ ;\sigma} = 0 \qquad or \qquad a_{a\ ;\sigma} = 0$$

which can be proven easily.
Using such parallel vector fields the changing rule easily leads back to the rule for a scalar. For the sake of simplicity, we perform the proof for a tensor T^λ with only one index. If ϕ is a scalar, the first thing that follows out of the definitions (16) and (21) is

$$\phi_{;\sigma;\tau} - \phi_{;\tau;\sigma} \equiv -\phi_{;\alpha}\ \Lambda_{\sigma\ \tau}^{\ \alpha}.$$

If we put the scalar $a_\lambda T^\lambda$ into this equation for ϕ, a_λ being a parallel vector field, a_λ may be interchanged with the differentiation symbol at every covariant differentiation, therefore a_λ appears as a factor in all the terms. Therefore, one obtains

$$[T^\lambda_{\ ;\sigma;\tau} - T^\lambda_{\ ;\tau;\sigma} + T^\lambda_{\ ;\alpha}\ \Lambda_{\sigma\ \tau}^{\ \alpha}]a_\lambda = 0.$$

This identity must hold for any choice of a_λ in a certain position, therefore the bracket vanishes, and we have finished our proof. The generalization for tensors with any number of Greek indices is obvious.
Identities for the tensor Λ. Permuting the indices κ, λ, μ in (20), adding the three identities, and by appropriate summing-up of the terms with respect to (21) one obtains

$$0 \equiv (\Lambda_{\kappa\ \lambda,\mu}^{\ \tau} + \Lambda_{\lambda\ \mu,\kappa}^{\ \tau} + \Lambda_{\mu\ \kappa,\lambda}^{\ \tau}) + \Delta_{\sigma\ \kappa}^{\ \tau}\Lambda_{\lambda\ \mu}^{\ \sigma} + \Delta_{\sigma\ \lambda}^{\ \tau}\Lambda_{\mu\ \kappa}^{\ \sigma} + \Delta_{\sigma\ \mu}^{\ \tau}\Lambda_{\kappa\ \lambda}^{\ \sigma}.^{6}$$

We convert this identity by introducing covariant instead of ordinary derivatives of the tensors a_λ (see (17)); so we acquire the identity

$$0 \equiv (\Lambda_{\kappa\ \lambda;\mu}^{\ \tau} + \Lambda_{\lambda\ \mu;\kappa}^{\ \tau} + \Lambda_{\mu\ \kappa;\lambda}^{\ \tau}) + (\Lambda_{\kappa\ \alpha}^{\ \tau}\Lambda_{\lambda\ \mu}^{\ \alpha} + \Lambda_{\lambda\ \alpha}^{\ \tau}\Lambda_{\mu\ \kappa}^{\ \alpha} + \Lambda_{\mu\ \alpha}^{\ \tau}\Lambda_{\kappa\ \lambda}^{\ \alpha}). \tag{24}$$

In order to express the a_λ's by the h in the above manner, this condition must be satisfied. Contraction of the above equation by the indices τ and μ yields the identity

$$0 \equiv \Lambda_{\kappa\ \lambda;\alpha}^{\ \alpha} + \phi_{\lambda;\kappa} - \phi_{\kappa;\lambda} - \phi_\alpha\Lambda_{\kappa\ \lambda}^{\ \alpha}.$$

[5] tr. note: cfr. Schouten III, (2.15)
[6] tr. note: cfr. Schouten III, (5.2)

or

$$\Lambda_{\kappa}{}^{\alpha}{}_{\lambda;\alpha} + \phi_{\lambda,\kappa} - \phi_{\kappa,\lambda}{}^{7} \tag{25}$$

where ϕ_λ stands for $\Lambda_{\lambda}{}^{\alpha}{}_{\alpha}$ (22).

4 The field equations

The most simple field equations we desired to find will be conditions for the tensor $\Lambda_{\lambda}{}^{\alpha}{}_{\nu}$. The number of h-components is n^2, of which n remain indeterminate due to general covariance; therefore the number of independent field equations must be $n^2 - n$. On the other hand, the higher number of possibilities a theory cuts down on (without contradicting experience), the more satisfactory it is. Therefore, the number Z of field equations should be as large as possible. If $\bar{Z}$ denotes the number of identities in-between the field equations, $Z - \bar{Z}$ must be equal to $n^2 - n$.
According to the change rule of differentiation

$$\Lambda_{\mu}{}^{\alpha}{}_{\nu;\nu;\alpha} - \Lambda_{\mu}{}^{\alpha}{}_{\nu;\alpha;\nu} - \Lambda_{\mu}{}^{\sigma}{}_{\tau;\alpha}\ \Lambda_{\sigma}{}^{\alpha}{}_{\tau} \equiv 0.^{8} \tag{26}$$

holds. An underlined index indicates "pulling up" and "pulling down" of an index, respectively, e.g.

$$\Lambda_{\underline{\mu}}{}^{\alpha}{}_{\underline{\nu}} \equiv \Lambda_{\beta}{}^{\alpha}{}_{\gamma}\ g^{\mu\beta}\ g^{\nu\gamma},$$

$$\Lambda_{\mu}{}^{\underline{\alpha}}{}_{\nu} \equiv \Lambda_{\mu}{}^{\beta}{}_{\nu}\ g_{\alpha\beta}.$$

We write this Identity (26) in the form

$$G^{\mu\alpha}{}_{;\alpha} - F^{\mu\nu}{}_{;\nu} + \Lambda_{\underline{\mu}}{}^{\sigma}{}_{\underline{\tau}} F_{\sigma\tau} \equiv 0, \tag{26a}$$

with the following settings

$$G^{\mu\alpha} \equiv \Lambda_{\underline{\mu}}{}^{\alpha}{}_{\underline{\nu};\nu} - \Lambda_{\underline{\mu}}{}^{\sigma}{}_{\underline{\tau}}\Lambda_{\sigma}{}^{\alpha}{}_{\tau}, \tag{27}$$

$$F^{\mu\nu} \equiv \Lambda_{\underline{\mu}}{}^{\alpha}{}_{\underline{\nu};\ \alpha}. \tag{28}$$

Now we make an ansatz for the field equations:

$$G^{\mu\alpha} = 0, \tag{29}$$

$$F^{\mu\alpha} = 0. \tag{30}$$

These equations seem to contain an forbidden overdetermination, because their number is $n^2 + \frac{n(n-1)}{2}$, while at first hand it is only known to satisfy the identities (26a).
Linking (25) with (30) it follows, that the ϕ_k can be derived from a potential. Therefore, we set

$$F_\kappa = \phi_\kappa - \frac{\partial \log \psi}{\partial x^\kappa} = 0. \tag{31}$$

(31) is completely equivalent with (30). The equations (29), (31) combined are $n^2 + n$ equations for $n^2 + 1$ functions $h_{s\nu}$ and ψ. Besides (26a) there is, however, another system of identities between

[7] tr. note: cfr. Schouten III, (5.6)
[8] tr. note: cfr. Schouten III, (4.9)

these equations we will derive now.
If $\underline{G}^{\mu\alpha}$ denotes the antisymmetric part of $G^{\mu\alpha}$, one can figure out directly from (27)

$$2\ \underline{G}^{\mu\alpha} = S_{\mu\ \alpha;\nu}^{\ \nu} + \frac{1}{2} S_{\underline{\sigma}\ \underline{\tau}}^{\ \mu} \Lambda_{\sigma\ \tau}^{\ \alpha} - \frac{1}{2} S_{\underline{\sigma}\ \underline{\tau}}^{\ \alpha} \Lambda_{\sigma\ \tau}^{\ \mu} + F^{\mu\alpha}, \tag{32}$$

For the sake of abbreviation we introduce the totally skew-symmetric tensor

$$S_{\underline{\mu}\ \underline{\nu}}^{\ \alpha} = \Lambda_{\underline{\mu}\ \underline{\nu}}^{\ \alpha} + \Lambda_{\underline{\alpha}\ \underline{\mu}}^{\ \nu} + \Lambda_{\underline{\nu}\ \underline{\alpha}}^{\ \mu}. \tag{33}$$

Figuring out the first term of (32) yields

$$2\ \underline{G}^{\mu\alpha} = S_{\underline{\mu}\ \underline{\alpha};\nu}^{\ \nu} - S_{\underline{\mu}\ \underline{\alpha}}^{\ \sigma} \Lambda_{\sigma\ \nu}^{\ \nu} + F^{\mu\alpha}, \tag{34}$$

But now, with respect to the definition of F_k (31)

$$\Delta_{\sigma\ \nu}^{\ \nu} - \Delta_{\nu\ \sigma}^{\ \nu} \equiv \Lambda_{\sigma\ \nu}^{\ \nu} \equiv \phi_\sigma \equiv F_\sigma + \frac{\partial \log \psi}{\partial x^\sigma}$$

or

$$\Delta_{\sigma\ \nu}^{\ \nu} = \frac{\partial \log \psi\ h}{\partial x^\sigma} + F_\sigma \tag{35}$$

holds. Therefore, (34) takes the form

$$h\psi(2\ \underline{G}^{\mu\alpha} - F^{\mu\alpha} + S_{\underline{\mu}\ \underline{\alpha}}^{\ \sigma} F_\sigma) \equiv \frac{\partial}{\partial x^\sigma}(h\ \psi S_{\underline{\mu}\ \underline{\alpha}}^{\ \sigma}) \tag{34b}$$

Due to the antisymmetry the desired system of identical equations follows

$$\frac{\partial}{\partial x^\alpha}[h\ \psi(2\ \underline{G}^{\mu\alpha} - F^{\mu\alpha} + S_{\underline{\mu}\ \underline{\alpha}}^{\ \sigma} F_\sigma)] \equiv 0 \tag{36}$$

These are at first n identities, but only $n-1$ of them are linear independent from each other. Because of the antisymmetry $[\ ]_{,\alpha,\mu} \equiv 0$ holds independently no matter what one inserts in $G^{\mu\alpha}$ and F_μ.
In the identities (4) and (36) you have to think of $F^{\mu\alpha}$ being expressed by F_μ according to the following relation which was derived from (31)

$$F_{\mu\alpha} \equiv F_{\mu,\alpha} - F_{\alpha,\mu}. \tag{31a}$$

Now we are able to prove the compatibility of the field equations (29), (30) or (29), (31), respectively.
First of all we have to show that the number of field equations minus the number of (independent) identities is smaller by n than the number of field variables. We have

number of field equations (29) (31) : $n^2 + n$
number of (independent) identities: $n + n - 1$
number of field variables: $n^2 + 1$,
$(n^2 + n) - (n + n - 1) = (n^2 + 1) - n$

As we see the number of identities just fits. We do not stop here, but prove the following
Proposition. *If in a cross section $x^n = const.$ all differential equations are satisfied and, in addition, $(n^2+1) - n$ of them are properly chosen everywhere, then all $n^2 + n$ equations are fulfilled anywhere.*
Proof.If all equations are fulfilled in the cross section $x^n = a$ and if these equations, that correspond to setting to zero the below, are fulfilled everywhere, we obtain:

$$
\begin{array}{ccc}
F_1 & \ldots & F_{n-1}F_n \\
G^{1\ 1} & \ldots & G^{1\ n-1} \\
\ldots & & \\
G^{n-1\ 1} & \ldots & G^{n-1\ n-1}.
\end{array}
$$

Then from (4) follows, that the $F^{\mu\alpha}$ vanish everywhere. Now one deduces from (36), that in an neighboring cross section $x^n = a + da$ the skew-symmetric $\underline{G}^{\mu\alpha}$ for $\alpha = n$ must vanish as well [9]. Out of (26a) it then follows, that in addition the symmetric $\underline{G}^{\mu\alpha}$ for $\alpha = n$ at the adjacent cross section $x^n = a + da$ must vanish. Repeating this kind of deduction proves the proposition.

5 First approximation

We are now going to deal with a field that shows very little difference from an Euclidean one with ordinary parallelism. Then we may set

$$h_{s\nu} = \delta_{s\nu} + \bar{h}_{s\nu}, \tag{37}$$

where $\bar{h}_{s\nu}$ is infinitely small at first order, higher order terms are neglected. Then, according to (5) and (6), we have to set

$$h_s{}^{\nu} = \delta_{s\nu} - \bar{h}_{\nu s}. \tag{38}$$

In first approximation, the field equations (29), (31) read

$$\bar{h}_{a\mu,\ \nu,\ \nu} - \bar{h}_{a\nu,\ \nu,\ \mu} = 0, \tag{39}$$

$$\bar{h}_{a\mu,\ a,\ \nu} - \bar{h}_{a\nu,\ a,\ \mu} = 0. \tag{40}$$

we substitute equation (31) by

$$\bar{h}_{a\nu,a} = \chi_{\nu}. \qquad (40a)$$

We claim now that there is an infinitesimal coordinate transformation $x^{\nu'} = x^{\nu} - \xi^{\nu}$, which causes all the variables $\bar{h}_{\alpha\nu,\nu}$ und $\bar{h}_{\alpha\nu,\alpha}$ to vanish.
Proof. First we prove that

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu}\ \xi^{\mu}{}_{,\nu}, \tag{41}$$

Therefore,

$$\bar{h}'_{a\nu,\nu} = h_{a\nu,\ \nu} + \xi^{\alpha}{}_{,\ \nu,\ \nu},$$

$$\bar{h}'_{a\nu a} = \bar{h}_{a\nu a} + \xi^{\alpha}{}_{,\ a\nu}.$$

The right sides vanish because of (40a), if the following equations are fulfilled

$$\begin{aligned} \xi^{\alpha}{}_{,\ \nu,\ \nu} &= -\bar{h}_{a\nu,\ \nu}, \\ \xi^{\alpha}{}_{,\ a} &= -\chi. \end{aligned} \tag{42}$$

[9] The $\frac{\partial G^{\mu n}}{\partial x^n}$ vanish for $x^n = a$.

But these $n+1$ equations for n variables ξ_α are compatible, because of (40a)

$$(-\bar{h}_{a\nu,\nu}),\ _a - (-\chi),\ _\nu,\ _\nu = 0.$$

Choosing new coordinates, the field equations read

$$\begin{aligned} \bar{h}_{a\mu,\ \nu\nu} &= 0, \\ \bar{h}_{a\mu,\ a} &= 0, \\ \bar{h}_{a\mu,\ \mu} &= 0, \end{aligned} \tag{43}$$

If we now separate $\bar{h}_{\alpha\nu,\nu}$ according to the equations

$$\begin{aligned} \bar{h}_{a\mu} + \bar{h}_{\mu a} &= \bar{g}_{a\mu}, \\ \bar{h}_{a\mu} - \bar{h}_{\mu a} &= a_{a\mu}, \end{aligned}$$

where $\delta_{\alpha\mu} + \bar{g}_{\alpha\mu}(= g_{\mu\nu})$ determines metrics in first approximation , thus the field equations take the simple form

$$\bar{g}_{a\mu,\ \sigma,\ \sigma} = 0, \tag{44}$$

$$\bar{g}_{a\mu,\ \mu} = 0, \tag{45}$$

$$a_{a\mu,\ \sigma,\ \sigma} = 0, \tag{46}$$

$$a_{a\mu,\ \mu} = 0. \tag{47}$$

One is led to suppose, that $\bar{g}_{\alpha\nu}$ and $a_{\alpha\mu}$ represent the gravitational and the electromagnetic field in first approximation respectively. (44), (45) correspond to Poisson's equation, (46), (47) to Maxwell's equations of the empty space. It is interesting that the field laws of gravitation seem to be separated from those of the electromagnetic field, a fact which is in agreement with the observed independence of the two fields. But in a strict sense none of them exists separately.
Regarding the covariance of the equations (44) to (47) we note the following. For the $h_{s\mu}$'s generally the transformation law

$$h'_{\ s\mu} = \alpha_{st} \frac{\partial x^\sigma}{\partial x^{\mu'}}\ h_{t\sigma}$$

holds. If the coordinate transformation is chosen linear and orthogonal as well as conform with respect to the twist of the local systems, that is

$$x^{\mu'} = \alpha_{\mu\sigma} x^\sigma, \tag{48}$$

we acquire the transformation law

$$h'_{s\mu} = \alpha_{st}\ \alpha_{\mu\sigma}\ h_{t\sigma}, \tag{49}$$

which is exactly the same as for tensors in special relativity. Because of (48) the same transformation law holds for the $\delta_{s\mu}$, so it also holds for the $\bar{h}_{\alpha\mu}, \bar{g}_{\alpha\mu}$, and $a_{\alpha\mu}$. With respect to such transformations the equations (44) to (47) are covariant.

6 Outlook

The big appeal of the theory exposed here, lies in its unifying structure and the high-level (but allowed) overdetermination of the field variables. I was able to show that the field equations yield equations, in first-order approximation, that correspond to the Newton-Poisson theory of gravitation and to Maxwell's theory of the electromagnetic field. Nevertheless I'm still far away from claiming the physical validity of the equations I derived. The reason for that is, that I did not succeed in deriving equations of motion for particles yet.